Calculus II (lecture #25?) Thurs Feb 21, 2013

§7.2

Def The ~~natural~~ logarithm is a function

~~$\log : (0,\infty) \to \mathbb{R}$~~ $\log : (0,\infty) \to \mathbb{R}$

given by

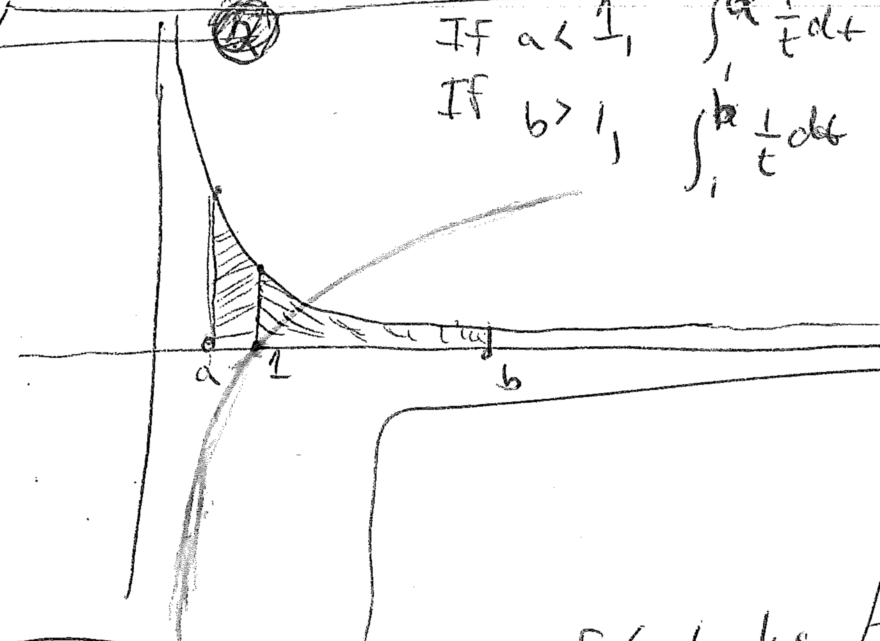
$$\log(x) = \int_1^x \frac{1}{t}\,dt$$

~~Comments~~: ~~(1)~~ Our book denotes this $\ln x$:

Note: $\ln x = \log x$

If $a < 1$, $\int_1^a \frac{1}{t}\,dt < 0$

If $b > 1$, $\int_1^b \frac{1}{t}\,dt > 0$



By the Chain Rule,

$$\frac{d}{dx}\log u = \frac{1}{u}\frac{du}{dx}$$

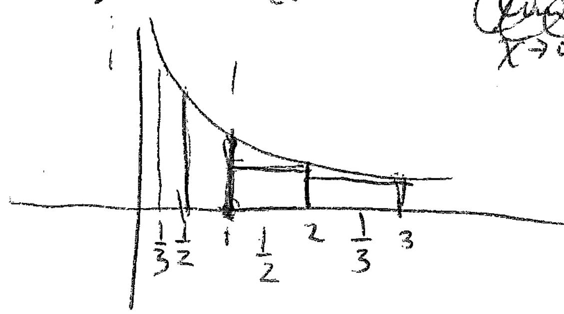
By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\log x = \frac{1}{x}.$$

~~Since $\frac{1}{x} > 0$ for $x > 0$,~~

~~log is increasing on $(0,\infty)$~~

$\boxed{\text{To find the}}$ domain and range of this function:



$$\log n > \sum_{k=2}^{n} \frac{1}{k}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{17} + \cdots + \frac{1}{32}\right)$$

$$> \underbrace{\frac{1}{2}}_{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}} + \frac{1}{2} + \frac{1}{2}$$

In general, if $x > 2^m$ some m,

$$\log x > \log 2^m > \sum_{k=2}^{2^m} \frac{1}{k} > \frac{m}{2}$$

So $\lim_{x \to \infty} \log x = \infty$

Similarly, $\lim_{x \to 0} \log x = -\infty$ (it blows up there)

So $\log x$ is not defined for $x < 0$

~~So dom(log)~~

and domain $\log = (0, \infty)$

range $\log = \mathbb{R}$

Facts

① $\frac{d}{dx} \log x = \frac{1}{x}$ $\cancel{\frac{d}{dx}\log ax = \frac{1}{x} \text{ for } a \in (0,\infty)}$

② $\log$ is increasing on $(0,\infty)$

③ $\log$ is injective

④ $\log 1 = 0$

⑤ $\frac{d}{dx} \log ax = \frac{1}{x}$ for all $a \in (0,\infty)$

⑥ $\log ax = \log a + \log x$ for $a, x \in (0,\infty)$

⑦ $\log x^r = r \log x$ for $x \in (0,\infty)$, $r \in \mathbb{Q}$.

⑧ $\lim_{x\to\infty} \log x = \infty$

$\lim_{x\to 0} \log x = -\infty$

⑨ $\log : (0,\infty) \to \mathbb{R}$ is bijective.

⑩ $\int \frac{1}{u}\,du = \log|u| + C$

Properties of log

1. domain $= (0, \infty)$
   range $= \mathbb{R}$
2. log is increasing, so it is injective. (3) $\frac{d}{dx} \log ax = \frac{1}{x}$ for $a \in (0,\infty)$
4. Product Rule: $\log ax = \log a + \log x$ for $a, x \in (0, \infty)$
5. Power Rule: $\log x^r = r \log x$ for $r \in \mathbb{Q}$

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pf (3) $\frac{d}{dx} \log ax = \cancel{\left(\frac{1}{ax}\right) a = \frac{1}{x}}$

$= \frac{dy}{dx}$ where $y = \log ax$

$= \frac{dy}{du}\frac{du}{dx}$ where $u = ax$, by chain rule

$= \left(\frac{1}{u}\right)(a) = \left(\frac{1}{ax}\right)(a) = \frac{1}{x}$

---

pf (4) Since $\log ax$ and $\log x$ have the same derivative, they differ by a constant ; for some constant C.

$\log ax = \log x + C$ so in particular if $x = 1$,

This is true for all x,

$\log a = \log 1 + C = 0 + C$

So $C = \log a$

So $\log ax = \log x + \log a$

pf 5) $\frac{d}{dx} \log x^r = \left(\frac{1}{x^r}\right)\left(r x^{r-1}\right) = \left(r \frac{1}{x}\right) = r \frac{d}{dx} \log x$

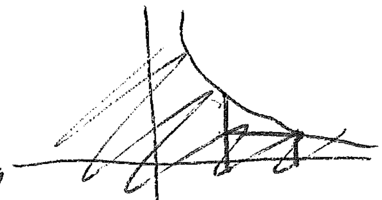
so $\log x^r$ and $r \log x$ differ by a constant

so $\log x^r = r \log x + C$ some C.

so $0 = \log 1^r = r \log 1 + C = C$ so $C = 0$

So $\log x^r = r \log x$

pf 8 ~~log x > log~~



Since $\frac{1}{x} > \frac{1}{2}$ on $[1,2]$,

$$\log 2 = \int_1^2 \frac{1}{t}\,dt > \int_1^2 \frac{1}{2}\,dt = \frac{1}{2}$$

So $\log 2^n = n\log 2 > \frac{n}{2}$ So $\lim_{x\to\infty} \log x = \infty$

~~So lim~~ Also, $\log 2^{-n} = -n\log 2 < -\frac{n}{2}$

~~Thus $\lim_{x\to\infty}$ lo~~ Thus $\lim_{x\to 0} \log x = -\infty$

---

pf 9 By 3, log is injective.
By 8, log is onto $\mathbb{R}$.

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pf 10 Since $\frac{d}{du}\log u = \frac{1}{u}$ for $x \in (0,\infty)$,
we have $\int \frac{1}{u}\,du = \log u + C = \log|u| + C$ for u a positive function.

If u is negative, $-u$ is positive, and

$$\int \frac{1}{(-u)}\,d(-u) = \log(-u) + C$$
$$= \log|u| + C$$

~~since u~~

Calculus II (Lecture # 25?) Friday Feb 22, 2013

HW §7.2 # 4, 9, 30, 43, 48, 62, 72

Recall:

Def $\log : (0,\infty) \to \mathbb{R}$ • increasing bijective

defined by $\log x = \int_1^x \frac{1}{t}\,dt$.

Properties ⓐ $\log 1 = 0$

ⓑ $\log ab = \log a + \log b$

ⓒ $\log a^r = r \log a$

~~[so if $r=-1$, $\log \frac{1}{a} = -\log a$]~~

~~We define the number e to be the~~

ⓓ $\log \frac{a}{b} = \log a - \log b$

[pf: $\log \frac{a}{b} = \log ab^{-1} = \log a + \log b^{-1}$
$= \log a - \log b$]

Also: $\frac{d}{dx} \log x = \frac{1}{x}$

$$\int \frac{1}{x}\,dx = \log|x| + C$$

Example

#26 Find $\frac{d}{d\theta} \log(\sec\theta + \tan\theta)$

$$= \left(\frac{1}{\sec\theta + \tan\theta}\right)\left(\frac{\sec\theta\tan\theta}{\cancel{\tan\theta\sec\theta}} + \sec^2\theta\right)$$

$$= \left(\frac{1}{\sec\theta + \tan\theta}\right)(\tan\theta + \sec\theta)\sec\theta$$

$$= \sec\theta$$

[conclude: $\int \sec\theta \, d\theta = \log(\sec\theta + \tan\theta) + C$

ϕ - ~~Find~~

#54 $\cancel{\int \sec\theta \, d\theta} = \cancel{\int \sec\theta \frac{\tan\theta + \sec\theta}{\sec\theta + \tan\theta} d\theta}$

$$\cancel{= \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} d\theta}$$

$$\int \frac{\sec x \, dx}{\sqrt{\log(\sec x + \tan x)}}$$

Let $u = \log(\sec x + \tan x)$ ~~Let $u = \sec\theta + \tan\theta$~~

So $du = \sec x \, dx$ ~~So $du = \sec\theta\tan\theta + \sec^2\theta$~~

$$= \int \frac{du}{\sqrt{u}}$$

$$= \int u^{-\frac{1}{2}} du$$

$$= \frac{u^{1/2}}{1/2} + C$$

$$= 2\sqrt{\log(\sec x + \tan x)} + C$$

Logarithmic Differentiation

(Avoid Product Rule)

Let $y = \dfrac{(x-2)^2(x^3+3x)}{x^4-1}$

Then $\log y = 2\log(x-2) + \log(x^3+3x) - \log(x^4-1)$

So $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2}{x-2} + \dfrac{3x^2+3}{x^3+3x} - \dfrac{4x^3}{x^4-1}$

So $\dfrac{dy}{dx} = \dfrac{(x-2)^2(x^3+3x)}{(x^4-1)}\left[\dfrac{2}{x-2} + \dfrac{3x^2+3}{x^3+3x} - \dfrac{4x^3}{x^4-1}\right]$

Example #64

$y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

So $\log y = \log\theta + \log\sin\theta - \frac{1}{2}\log\sec\theta$

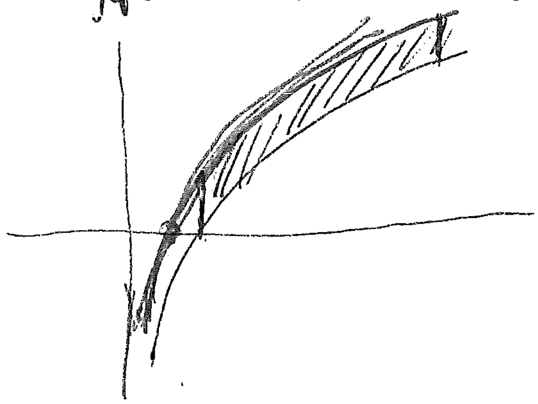
So $\dfrac{1}{y}\dfrac{dy}{d\theta} = \dfrac{1}{\theta} + \dfrac{1}{\sin\theta}(\cos\theta) - \dfrac{1}{2}\dfrac{1}{\sec\theta}\sec\theta\tan\theta$

$= \dfrac{1}{\theta} + \cot\theta - \dfrac{\tan\theta}{2}$

So $\dfrac{dy}{d\theta} = \left(\dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}\right)\left(\dfrac{1}{\theta} + \cot\theta - \dfrac{\tan\theta}{2}\right)$

§7.2 #71

Find the area between $y = \log x$ and $y = \log 2x$.
from $x = 1$ to $x = 5$



Note:
$\log 2x = \log x + \log 2$
So the graph of $\log 2x$
is the graph of $\log x$
shifted up by $\log 2$.

So we expect to get $4(\log 2)$ by Cavallieri's Principle.

Compute:

$$\int_1^5 (\log 2x - \log x)\,dx = \int_1^5 (\log x + \log 2 - \log x)\,dx$$

$$= \int_1^5 \log 2 \,dx$$

$$= (\log 2)x \Big|_1^5 = 4 \log 2$$

$$= \log 16$$

Calculus II (Lesson #27?) Feb 2013

Recall that ~~..~~ we defined

~~$\log : (0,\infty) \to \mathbb{R}$~~ the natural logarithm

is ~~defined by~~ $\log x = \int_1^x \frac{1}{t}\,dt$

so that $\frac{d}{dx} \log x = \frac{1}{x}$ and $\int \frac{1}{x}\,dx = \log|x| + C$.

We showed that

$$\log : (0,\infty) \to \mathbb{R} \text{ is bijective.}$$

Thus, log has an inverse function.

We define ~~..~~ the ~~expon~~ natural exponential function

$$\exp : \mathbb{R} \to (0,\infty)$$

to be the inverse function; $\exp = \log^{-1}$,

so that ~~log~~ $\exp(\log(x)) = x$

and $\log(\exp(x)) = x$

②

Properties of exp

① $\exp(a+b) = \exp(a)\exp(b)$

② $(\exp(a))^b = \exp(ab)$

---

pf ① ~~Since log is onto $\mathbb{R}$,~~

~~$\exists\, c, d \in (0,\infty)$ such that $a = \log c$ and $b = \log d$~~

~~So $\exp(a)\exp(b) = cd$~~

~~So $\exp(a+b) = \exp(\log c + \log d)$~~

~~$= \exp(\log cd)$~~

~~$= cd$~~

~~$= \log$~~

Let

$c = \exp(a)$

$d = \exp(b)$.

Then $\log c = a$ and $\log d = b$.

~~Now~~ Thus $\exp(a+b) = \exp(\log c + \log d)$

$= \exp(\log cd)$

$= cd$

$= \exp(a) + \exp(b)$

~~Also, $(\exp(a))^b = c^b$~~

Also, $\log \exp(a)^b = b\log(\exp(a)) = ba = ab$

Take exp of both sides to get

So $\exp(a)^b = \exp(ab)$

~~Since exp is surjective onto $(0,\infty)$~~,

~~then~~ We define the number e by

$$e = \exp(1).$$

This is an irrational number.

~~Note $\log e = 1$.~~

Prop if $\alpha \in \mathbb{Q}$, $e^{\alpha} = \exp(\alpha)$

~~Pf: $\alpha = \frac{a}{b}$ some $a, b \in \mathbb{Z}$~~

$\cancel{e^{\alpha}}$ e^{a}

Pf $\log e^{\alpha} = \alpha \log e = \alpha$

Take exp of both sides

So $e^{\alpha} = \exp \alpha$

Examples

(a) $\log e = 1$

(b) $\log e^2 = 2 \log e = 2$

(c) $\log e^x = x \log e = x$

~~(d) $e^{\log x} = x$~~ (d) $e^{\log 2} = 2$

~~(e) $e^{\log 4} = e$~~ (e) $e^{3 \log 2} = e^{\log 2^3} = e^{\log 8} = 8.$

~~We are now in a position~~

Recall

$$a^n = \underbrace{a \cdots a}_{n \text{ times}}$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

We are now in a position to define a^x for x any arbitrary real number, if $a > 0$.

<u>Def</u> Let $a > 0$ and $x \in \mathbb{R}$

Set $$a^x = \exp(x \log(a)).$$

Note: if x is rational,
$$\exp(x \log(a)) = \exp(\log(a^x)) = a^x,$$
so this agrees with our previous definition.

~~Thm (Rules of exponentiation).~~ Note:

① if we fix a and let x vary through the reals, we obtain a function

$$\exp_a : \mathbb{R} \to (0,\infty)$$

given by $\exp_a(x) = a^x$.

~~In particular, $\exp_e(x) = e^x = \exp x$~~

~~whose graph is~~

Theorem

$$e = \lim_{x \to 0} (1+x)^{1/x}$$

Pf Let $L = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$.

Then $\log L = \log \lim_{x \to 0} (1+x)^{\frac{1}{x}}$

$$= \lim_{x \to 0} \log (1+x)^{\frac{1}{x}}$$

$$= \lim_{x \to 0} \frac{1}{x} \log(1+x)$$

$$= \lim_{h \to 0} \frac{1}{h} \log(1+h)$$

$$= \lim_{h \to 0} \frac{\log(1+h) - \log(1)}{h}$$

$$= \log'(1)$$

$$= \frac{1}{1} = 1$$

So $\log L = 1 \Rightarrow L = \exp(1) = e$

HW
§7.3 #10, 24, 35
43, 50, 68
§7.4 #7, 14, 52, 62, 33

Calculus II (Lesson #29?) Thurs Feb 28, 2013

Recall $\log : (0,\infty) \to \mathbb{R}$ is bijective
$\exp : \mathbb{R} \to (0,\infty)$ is its inverse.

$e = \exp(1)$
$e^x = \exp(x)$ [since $\exp(a)^b = \exp(ab)$]

$\frac{d}{dx} e^x = e^x$ So $\int e^x \, dx = e^x + C$

Example $\frac{d}{dx} e^{x^2} = 2x e^{x^2}$

#53 $\int \frac{e^{1/x}}{x^2} dx$

Let $u = \frac{1}{x}$
So $du = -\frac{1}{x^2}$

$= -\int e^u \, du = -e^u + C = -e^{\frac{1}{x}} + C$

Power Rule (General)

We know that for $r \in \mathbb{Q}$, $\frac{d}{dx} x^r = r x^{r-1}$.

Now let $\alpha \in \mathbb{R}$ be any real number.

Then $x^\alpha = \exp(\alpha \log(x))$.

So $\frac{d}{dx} x^\alpha = \frac{d}{dx} \exp(\alpha \log(x)) = \exp(\alpha \log x)\left[\alpha \frac{1}{x}\right]$

$= x^\alpha \left[\alpha \frac{1}{x}\right]$

$= \alpha x^{\alpha - 1}$

So this proves the general case.

~~Sine~~ Since ~~$\exp_a$ is injective~~

Def , $\exp_a : \mathbb{R} \to (0,\infty)$ is bijective, it has an inverse.

~~Define the~~ The logarithm base a ~~to be~~ is the function

$$\log_a : (0,\infty) \to \mathbb{R}$$

~~to~~ defined to be the inverse of $\exp_a$.

So $\log_a a^x = x$ and $a^{\log_a x} = x$.

Facts

① $\log_a 1 = 0$

② $\log_a a = 1$

Examples

~~ⓐ $\log_2$ ⓐ $\log_2 32 = \log_2 2^5 = 5\log_2 2 = 5$~~

and ~~So~~, $\log_a x = y \iff a^y = x$

Facts ① $\log_a 1 = 0$

② $\log_a a = 1$

③ $\log_a xy = \log_a x + \log_a y$

④ $\log_a x^r = r\log_a x \quad r \in \mathbb{R}$

Examples ⓐ $\log_2 32 = \log_2 2^5 = 5\log_2 2 = 5$.

ⓑ $2^{\log_2 3} = 3$

ⓒ $\log_{10} 10000 = \log_{10} 10^4 = 4$ (= # zeros)

<u>Recall</u> $\exp_a : \mathbb{R} \to (0, \infty)$

Given by $\exp_a x = a^x = \exp(x \log(a))$.

$$\cancel{\frac{d}{dx} \exp_a x = \frac{d}{dx} a^x = \frac{d}{dx}}$$

So $\frac{d}{dx} a^x = \frac{d}{dx} \exp_a x = \frac{d}{dx} \exp(\log(a)\, x)$

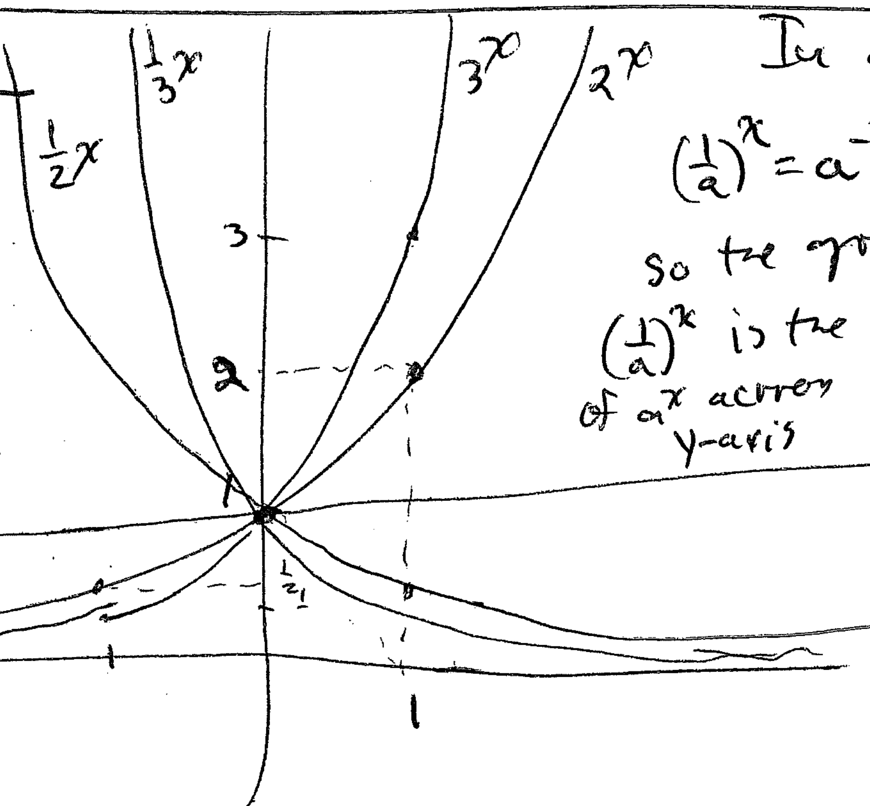
$$= \exp(\log(a)\, x) \log(a)$$

$$= \log(a)\, a^x.$$

So $\int a^x \, dx = \frac{a^x}{\log(a)} + C$.

<u>Exponential Functions</u>

a^x



In general

$\left(\frac{1}{a}\right)^x = a^{-x}$

so the graph of $\left(\frac{1}{a}\right)^x$ is the reflection of a^x across y-axis

a^x

Note

$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$

How logs are related:

Let $y = \log_a x$

Then $a^y = x$

So $y \log a = \log x$

So $y = \frac{\log x}{\log a}$ i.e. $\boxed{\log_a x = \frac{\log x}{\log a}}$

$$\boxed{\frac{d}{dx} \log_a x = \frac{d}{dx} \frac{\log x}{\log a} = \frac{1}{(\log a)x}}$$